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CALCULATION OF MECHANICAL SYSTEMS WITH PULSED EXCITATION†

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It is suggested that the solutions of differential equations for linear systems with periodic impulsive excitation can be found in a special form which contains a standard pair of non-smooth periodic functions and possesses the structure of an algebra without division. This form is also suitable in the case of excitation with a periodic series of discontinuities of the first kind. © 1996 Elsevier Science Ltd. All rights reserved.

The action of instantaneous impulses on a mechanical system is customarily simulated using one of the two methods: by subjecting the coordinates and velocities to additional conditions in the neighbourhoods of the points of localization of the impulses such as, for example, by specifying velocity jumps at the instants of external impacts or by introducing singular terms of the Dirac δ-function type into the equations.

The principal merit of the first approach is the fact that the differential equations describing the system are the same as when there are no impulses acting [1]. However, these equations are treated separately in each of the intervals between the impulses, and hence, instead of a single system, a whole sequence of systems is analysed. The second method gives a single system of equations over the whole time interval without introducing the above-mentioned conditions imposed on the variables, but the analysis can be carried out correctly within the framework of the theory of generalized functions (distributions) [2, 3], which requires additional mathematical proofs in non-linear cases [4].

A method is described in this paper which enables one, on the one hand, to eliminate the singular terms in the equations and, on the other hand, to obtain solutions, by analysing a boundary-value problem in a standard interval, in the form of a single analytic expression over the whole time interval.

In this connection, we note the paper by Liu Zheng-rong [5] where the solutions of quasilinear systems $x(t)$ under the action of an impulsive or discontinuous force (with a singularity at the point $t = a$) are found in the form

$$x(t) = x_1(t) + x_2(t)H(t-a) \tag{0.1}$$

where $H(\cdot)$ is the Heaviside step function.

An analogous (0.1) form of the solution is used to describe moving discontinuities in wave theory in [6].

In the case of systems with rigid constraints, special (non-smooth) transformations have been constructed [7, 8] which instantaneously rotate the coordinate axes at the instants of impact on a system without an arresting device. As a result of the transformation, the system is freed from its bonds and the corresponding differential equations do not contain any "impact" terms whatsoever. The representation used below for the solution instantaneously changes the direction and scale of time but not of the spatial coordinates at the instants the external impulses act. After introducing an appropriate time parameter, the spatial variable acquires the form of an element of an algebra. It is subsequently shown that this enables one easily to use a new spatial variable in the transformations associated with the solution of the differential equations.

The relations described below were obtained earlier for the symmetric case [9, 10] ($\gamma = 0$; Fig. 1) and have been used to calculate the oscillations of strongly non-linear mechanical systems [11]. The problem of generalizing the relations to take account of asymmetry ($\gamma \neq 0$) has been formulated by Starushenko when investigating the statics of elastic systems having a periodically inhomogeneous structure.

1. PIECEWISE-LINEAR PERIODIC ARGUMENT

Let $\tau(\varphi; \gamma)$ be a sawtooth piecewise-smooth function of argument φ with a single amplitude and period equal to four (such a normalization of the period is convenient for the subsequent transformations)

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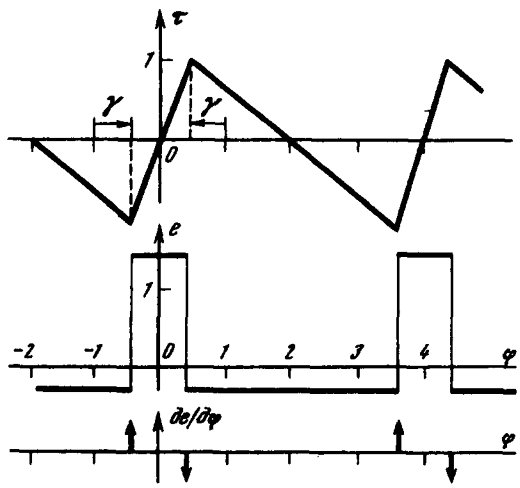

Fig. 1.

$$\tau(\varphi;\gamma)=\begin{cases}\varphi/(1-\gamma), & -1+\gamma\leqslant\varphi\leqslant 1-\gamma\\ (-\varphi+2)/(1+\gamma), & 1-\gamma\leqslant\varphi\leqslant 3+\gamma\end{cases} \tag{1.1}$$

$$\tau(\varphi+4;\gamma)\equiv\tau(\varphi;\gamma)$$

where γ $(-1 < \gamma < 1)$ is a parameter which characterizes the slope of the "saw" (Fig .1). When $\gamma = 0$, the "saw" becomes symmetric.

We will denote a generalized derivative with respect to the argument φ by $e(\varphi; \gamma) = \partial\tau(\varphi; \gamma)/\partial\varphi$ (the notation used here is associated with the role of the functions τ and e in the context of this paper: τ is subsequently a new time variable and e is a basis element of an algebra).

Proposition 1. Any periodic function $x(\varphi)$ with a period $T = 4$ can be represented in the form

$$x = X(\tau) + Y(\tau)e;\quad \tau = \tau(\varphi; \gamma),\quad e = e(\varphi; \gamma) \tag{1.2}$$

Proof. We shall define $X(\tau)$, $Y(\tau)$ as

$$X=\frac{1}{2\alpha}\left\{\frac{1}{1+\gamma}x[(1-\gamma)\tau]+\frac{1}{1-\gamma}x[2-(1+\gamma)\tau]\right\}$$

$$Y=\frac{1}{2\alpha}\{x[(1-\gamma)\tau]-x[2-(1+\gamma)\tau]\},\quad \alpha=\frac{1}{1-\gamma^2}$$

Then, in a period, we have
when $-1 + \gamma \leqslant \varphi \leqslant 1 - \gamma$

$$x(\varphi)=X\left(\frac{\varphi}{1-\gamma}\right)+Y\left(\frac{\varphi}{1-\gamma}\right)\frac{1}{1-\gamma}\equiv x(\varphi)$$

and, when $1 - \gamma \leqslant \varphi \leqslant 3 + \gamma$

$$x(\varphi)=X\left(\frac{-\varphi+2}{1+\gamma}\right)-Y\left(\frac{-\varphi+2}{1+\gamma}\right)\frac{1}{1+\gamma}\equiv x(\varphi)$$

Remark 1. If the function $x(\varphi)$ has a period of $4a$, we must make the substitution

$$\tau(\varphi;\gamma) \to a\tau(\varphi/a;\gamma)$$

in the expressions for X and Y.

Proposition 2. The elements (1.2) possess an algebraic structure (with 1 and e as the basis elements of the algebra) such that the relation

$$f(X+Ye) = R_f + I_f e \tag{1.3}$$

$$R_f = \frac{1}{2\alpha}\left[\frac{1}{1+\gamma}f(Z_+) + \frac{1}{1-\gamma}f(Z_-)\right]$$

$$I_f = \frac{1}{2\alpha}\left[f(Z_+) - f(Z_-)\right], \quad Z_\pm = X \pm \frac{Y}{1\mp\gamma}$$

holds for any regular function $f(x)$.

Proof. These relations can be immediately verified since either $e = 1/(1-\gamma)$ or $e = -1/(1-\gamma)$ almost everywhere with respect to φ. The "multiplication table" of this algebra is

$$e^2 = \alpha + \beta e, \quad \beta = 2\gamma\alpha \tag{1.4}$$

In the case of a symmetric saw, $\gamma = 0$, Eq. (1.4) acquires the simplest form ($e^2 = 1$) and the elements (1.2) form an algebra of hyperbolic numbers [12]. This version of the representation of periodic solutions has been used previously in [9–11].

Proposition 3. The result of differentiating the representation (1.2) remains in the algebra under consideration in the case of a continuous function $x(\varphi)$.

Proof. This fact is obvious since the derivative of a continuous periodic function is a function which satisfies the conditions of Proposition 1. However, for the subsequent discussion, we shall give the corresponding expression for the derivative

$$dx/d\varphi = \alpha Y' + (X' + \beta Y')e + Y\partial e/\partial\varphi$$

where a prime denotes differentiation with respect to τ. Since the initial expression (1.2) contains the function $e = e(\varphi)$ with singularities of the first kind at the points $\Lambda = \{\varphi \cdot \tau(\varphi) = \pm 1\}$, the result of differentiation formally contains a periodic singular term (the last term on the right-hand side). In the case of a continuous function $x(\varphi)$, this term is discarded since δ-pulses

$$\frac{\partial e}{\partial\varphi} = 2\alpha\sum_{k=-\infty}^{\infty}[\delta(\varphi+1-\gamma-4k) - \delta(\varphi-1+\gamma-4k)] \tag{1.5}$$

are "concentrated" at points where the Y-component of the function $x(\varphi)$ vanishes: $Y|_{\varphi\in\Lambda} = Y|_{\tau=\pm 1} = 0$. If, however, discontinuities of the first kind are permitted at the points Λ in the case of the initial function $x(\varphi)$, it is possible, as will be shown later, to dispose of the singular term which has appeared with the aim of eliminating singular terms from the equations of motion.

Proposition 4. When

$$\int_{-1}^{1} X(\tau)d\varphi = 0$$

the result of integrating expression (1.2) remains in the algebra considered

$$\int(X+Ye)d\varphi = Q(\tau) + P(\tau)e$$

$$Q = \int\left[Y(\tau) - \frac{\beta}{\alpha}X(\tau)\right]d\tau, \quad P = \frac{1}{\alpha}\int_{-1}^{\tau}X(\xi)d\xi$$

Proof. The equality can be verified by differentiation with respect to the variable φ.

2. A SYSTEM WITH PERIODIC PULSED EXCITATION

We will now consider a mechanical system with periodic pulsed excitation, described by the equation

$$\dot{x} = f(x, \varphi) + p\partial e/\partial\varphi; \quad x \in R^n, \quad \varphi = \omega t \tag{2.1}$$

where the dot denotes differentiation with respect to the variable t, the period of the right-hand side with respect to the variable φ is equal to four, γ $(-1 < \gamma < 1)$ is a given parameter, p is a constant n-dimensional vector, and the vector function f is a regular component of the right-hand side and is assumed to be continuous with respect to each of the set of variables x and is piecewise continuous with respect to φ: discontinuities of the first kind are permitted at the points where the periodically acting δ-impulses $\partial e/\partial\varphi$ (1.5) are localized

The above assumptions enable us to consider Eq. (2.1) in the sense of a distribution (with respect to the variable t).

We shall seek a periodic solution of Eq. (2.1) in the form

$$x = X(\tau) + Y(\tau)e; \quad \tau = \tau(\varphi; \gamma), \quad e = e(\varphi; \gamma) \tag{2.2}$$

where X and Y are vector functions, to be determined. The idea of representing the required function in the form of (2.2) is based on the possibility of representing any periodic function in the form of (1.2) (Proposition 1).

Substituting (2.2) into (2.1) and taking account of relations (1.2)–(1.4), we obtain

$$\omega\alpha Y' - R_f + [\omega(X' + \beta Y') - I_f]e + (Y\omega - p)\partial e/\partial\varphi = 0 \tag{2.3}$$

$$R_f = \frac{1}{2\alpha}\left\{\frac{1}{1+\gamma}f[Z_+,(1-\gamma)\tau] + \frac{1}{1-\gamma}f[Z_-,2-(1+\gamma)\tau]\right\}$$

$$I_f = \frac{1}{2\alpha}\{f[Z_+,(1-\gamma)\tau] - f[Z_-,2-(1+\gamma)\tau]\}$$

Eliminating the last term (the periodically singular term) on the left-hand side of Eq. (2.3) by means of the conditions imposed on Y, which is a component of the required solution

$$Y|_{\tau=\pm1} = p/\omega \tag{2.4}$$

we note that the remaining terms form an element of the algebra being considered. On equating the coefficients of the basis elements separately to zero, we obtain the system of equations

$$\omega\alpha Y' = R_f, \quad \omega(X' + \beta Y') = I_f \tag{2.5}$$

Hence, we have changed from the initial system with a periodic impulsive action to the boundary-value problem (2.4), (2.5) in the standard interval $-1 \leqslant \tau \leqslant 1$ which does not contain any singular terms. If $p = 0$, that is, there is no singular periodic component in the initial equation, the boundary conditions become homogeneous.

Example. Consider the linear system

$$\dot{x} = Mx + p\partial e/\partial\varphi, \quad \varphi = \omega t$$

where M is a constant non-singular $n \times n$ matrix which is an orthonormalized system of eigenvectors e_j

$$Me_j = \lambda_j e_j \quad (j = 1,\ldots, n)$$

In this case, Eqs (2.5) take the form

$$X = \omega M^{-1}Y', \quad \omega(X' + \beta Y') = MY \tag{2.6}$$

Eliminating the vector X from the second equation using the first equation, we obtain the equation

$$\omega^2 Y'' + \omega \frac{\beta}{\alpha} MY' - \frac{1}{\alpha} M^2 Y = 0$$

the solution of which, subject to the boundary conditions (2.4), we find in the form

$$Y = \frac{1}{\omega} \sum_{j=1}^{n} p^j e_j \left[\operatorname{ch}(\gamma\zeta_j) \frac{\operatorname{ch}(\zeta_j)}{\operatorname{ch}\zeta_j} + \operatorname{sh}(\gamma\zeta_j) \frac{\operatorname{sh}(\zeta_j)}{\operatorname{sh}\zeta_j} \right] \exp(-\gamma\zeta_j\tau), \quad \zeta_j = \frac{\lambda_j}{\omega} \tag{2.7}$$

where $p^j = e_j^T p$ are the coordinates of the vector p in the basis $\{e_1, \ldots, e_n\}$ and the first equality of (2.6) gives X, a component of the solution.

3. PARAMETRIC EXCITATION

We will now consider the case of parametric excitation taking the example of a linear system described by the equation

$$\ddot{x} + [q(\varphi) + p\partial e/\partial\varphi]x = 0; \quad x \in R^n, \quad \varphi = \omega t \tag{3.1}$$

where p is a constant $n \times n$ matrix and $q(\varphi)$ is a periodic (with period $T = 4$) piecewise-continuous matrix function of the same dimension (discontinuities of the first kind at the points Λ of the action of impulses are allowed).

The regular component of the parametric excitation is represented in the form

$$q(\varphi) = Q(\tau) + P(\tau)e \tag{3.2}$$

and we shall seek a solution of Eq. (3.1) which is periodic with period 4 with respect to the variable $\varphi = \omega t$ in the form of (2.2).

Substituting (2.2) and (3.2) into Eq. (3.1) and bearing in mind the equality $e^2 = \alpha + \beta e$, with the necessary condition of continuity of the vector function $x(t)$

$$Y|_{\tau=\pm 1} = 0 \tag{3.3}$$

we obtain an equality which relates e, $\partial e/\partial\varphi$, $e\partial e/\partial\varphi$.

Since the relation

$$e\frac{\partial e}{\partial \varphi} = \frac{1}{2}\beta\frac{\partial e}{\partial \varphi} \tag{3.4}$$

holds in the sense of a distribution, using (3.3), we obtain

$$\omega^2(\alpha X'' + \alpha\beta Y'') + QX + \alpha PY = 0 \tag{3.5}$$

$$\begin{gathered} \omega^2[(\alpha + \beta^2)Y'' + \beta X''] + PX + QY + \beta PY = 0 \\ [\omega^2(X' + \beta Y') + pX]|_{\tau=\pm 1} = 0 \end{gathered} \tag{3.6}$$

from the above-mentioned equality.

Together with (3.3), these relations form a boundary-value problem for determining the vector functions X and Y and the corresponding relations between the parameters for which periodic solutions exist.

Note that, in the case of impulses of constant direction in condition (3.6), it is necessary to substitute $\pm p$ or $\mp p$ instead of p depending on the direction of the action of the impulses.

We shall show that Eq. (3.4), obtained by formal differentiation of both sides of Eq. (1.4), holds in the sense of a distribution. In the theory of generalized functions, the product of a Dirac δ-function and a function which has a discontinuity at a "point of localization" of a δ-impulse does not have a definite meaning in the general case

[2–4]. Since, in fact, such functions with singularities at the points $\Lambda = \{t: \tau(\omega t) = \pm 1\}$ are also multiplied on the left-hand side of equality (3.4), we shall consider the latter from the point of view of the theory of generalized functions locally, in the neighbourhood of one of the points of the set Λ and, actually, $t = 1 - \gamma$ (we assume that $\omega = 1$).

The possibility of assigning definite meaning to the left-hand side of (3.4) is attributable, in the final analysis, to the fact that both factors are generated by the one and the same limiting sequence of smooth functions. So, let $\omega_\varepsilon(\varphi)$ be a sequence of smooth functions (of "caps" [2]) which are equal to zero outside the interval $-\varepsilon < \varphi < \varepsilon$ such that

$$\int_{-\varepsilon}^{\varepsilon} \omega_\varepsilon(\varphi) d\varphi = 1 \qquad \text{for all } \varepsilon$$

We have that $\omega_\varepsilon(\varphi) \to \delta(\varphi)$ when $\varepsilon \to 0$ in the sense of a weak limit. The sequences of smooth functions which approximate the functions e and e' in the neighbourhood of the point $t = 1 - \gamma$ can be chosen in the following way

$$e_\varepsilon = \frac{1}{1-\gamma} - \beta\theta_\varepsilon(t-1+\gamma), \quad \frac{\partial e_\varepsilon}{\partial\varphi} = -\beta\omega_\varepsilon(t-1+\gamma); \quad \theta_\varepsilon(\varphi) = \int_{-\infty}^{\varphi} \omega_\varepsilon(\xi)d\xi \tag{3.7}$$

$$(-1 + \gamma < t < 3 + \gamma)$$

In the sense of a weak limit, we have

$$e_\varepsilon \to e, \ \partial e_\varepsilon/\partial\varphi \to \partial e/\partial\varphi \ \text{ when } \varepsilon \to 0$$
$$(-1 + \gamma < t < 3 + \gamma)$$

We now consider the limiting equality (3.4). Replacing e, e' by e_ε, e'_ε, we obtain after reduction

$$\theta_\varepsilon\omega_\varepsilon = \omega_\varepsilon/2 \tag{3.8}$$

For simplicity, we will now move the origin to the point $t = 1 - \gamma$ and show that the left-hand side of the equality, when $\varepsilon \to 0$, gives $\delta(t)/2$ in the sense of a weak limit [4], that is, the same as the right-hand side.

Note that (henceforth integration is carried out everywhere from $-\varepsilon$ to ε)

$$\int \theta_\varepsilon(t)\omega_\varepsilon(t)dt = \int \theta_\varepsilon(d\theta_\varepsilon/dt)dt = \theta_\varepsilon^2/2\big|_{-\varepsilon}^{\varepsilon} = \tfrac{1}{2} \tag{3.9}$$

Let $\phi(t)$ be a continuous trial function. In an ε-neighbourhood of the point $t = 0$, we have $|\phi(t) - \phi(0)| < 2\eta$. Hence, when account is taken of (3.9), we obtain the limit

$$\left|\int \theta_\varepsilon(t)\omega_\varepsilon(t)\phi(t)dt - \phi(0)/2\right| = \leqslant \int \theta_\varepsilon(t)\omega_\varepsilon(t)\left|\phi(t) - \phi(0)\right|dt \leqslant \eta$$

This also means that

$$\int \theta_\varepsilon(t)\omega_\varepsilon(t)\phi(t)dt \to \phi(0)/2 \text{ when } \varepsilon \to 0$$

Without loss of generality of the transformations we shall henceforth assume that $\omega = 1$ ($\varphi \equiv t$). We can always achieve this by means of a scale transformation of the variables.

Example. As an example, we shall consider the case of a system with a single degree of freedom. Let $P, Q =$ const. Equation (3.1) then takes the form

$$\ddot{x} + [Q + Pe + p\partial e/\partial t]x = 0; \quad e = e(t; \gamma), \quad x \in R \tag{3.10}$$

We shall study the relation between the parameters of the regular (piecewise-linear) impulsive components of the action for which the periodic oscillations of the system, with a period equal to the period of the action, are possible.

We put

$$x = X + Ye; \ X = B\exp(\lambda\tau), \ Y = B\mu\exp(\lambda\tau); \ B, \mu, \lambda = \text{const.}$$

From the differential equations for X and Y (3.5), subject to the condition $B \neq 0$, we obtain a characteristic equation which has the following two pairs of roots

$$\lambda^2 = -(1-\gamma)P - (1-\gamma)^2 Q \equiv \pm k^2, \quad \lambda^2 = (1+\gamma)P - (1+\gamma)^2 Q \equiv \pm l^2$$

We now consider the case of negative pairs. This means that the regular part of the coefficient of x in Eq. (3.10) is positive

$$Q + Pe(t;\gamma) > 0 \quad \text{for all} \quad t$$

The general solution of the system of equations subject to this condition will have the form

$$X = B_1 \sin k\tau + B_2 \cos k\tau + B_3 \sin l\tau + B_4 \cos l\tau$$

$$Y = \mu_1 (B_1 \sin k\tau + B_2 \cos k\tau) + \mu_2 (B_3 \sin l\tau + B_4 \cos l\tau)$$

$$\mu_1 = -\frac{1}{\alpha}\frac{-\alpha k^2 + Q}{-\beta k^2 + P}, \quad \mu_2 = -\frac{1}{\alpha}\frac{-\alpha l^2 + Q}{-\beta l^2 + P}$$

In view of the homogeneity of the boundary conditions, the requirement that the solution should be non-trivial leads to the following relation $p = p(P, Q; \gamma)$

$$\begin{aligned} p^2 = &-\{[2(k^2\mu_1\mu_1^2 + l^2\mu_1^2\mu_2)\beta + (k^2 + l^2)\beta^2\mu_1^2\mu_2^2 + k^2\mu_2^2 + l^2\mu_1^2]\times \\ &\times \frac{1}{4}\sin 2k \sin 2l - [(\mu_1^2\mu_2 + \mu_1\mu_2^2)\beta + \beta^2\mu_1^2\mu_2^2 + \mu_1\mu_2]\times \\ &\times [\cos^2 k \sin^2 l + \cos^2 l \sin^2 k]kl\}[(\mu_1 - \mu_2)^2 \frac{1}{4}\sin 2k \sin 2l]^{-1} \end{aligned} \tag{3.11}$$

The dependence of p on Q (for fixed P and γ) has a branched (zone) structure characteristic of similar problems (see Fig. 2, where, in view of the symmetry, only the upper half-plane is shown. The calculations were carried out with the value of $P = 10^{-4}$). A curious effect is observed when the parameter γ ($-1 < \gamma < 1$) which characterizes the slope of the "saw" is varied. The question concerns the transverse "collapse" of the sequences of zones which are regularly situated across each of the s zones, where s is an integer which depends on the parameter γ (the form of this relationship will be established later). For instance, when $\gamma = 1/5$, each fifth zone is missing and, when $\gamma = 1/2$, each second zone is missing (see Fig. 2a and b, respectively).

In order to investigate this effect, we shall consider the case $P = 0$ when the dependence presented above takes the following simpler form

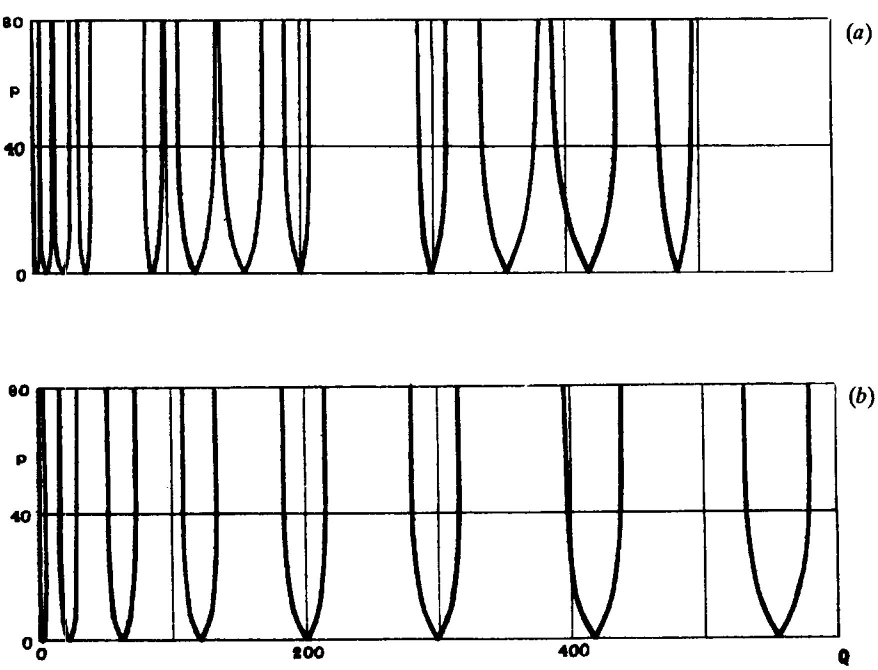

Fig. 2.

$$p^2 = -\frac{Q}{\alpha^2}\frac{\sin^2(k+l)}{\sin 2k \sin 2l}; \quad k = (1-\gamma)Q^{1/2}, \quad l = (1+\gamma)Q^{1/2} \tag{3.12}$$

We shall analyse this expression. The numerator vanishes for the values

$$Q = (j\pi/2)^2, \quad j = 0, 1, 2, \ldots \tag{3.13}$$

Branches of the dependence of p on Q (zones) start out from these points on the OQ axis. However, if the slope of the "saw" parameter γ is rational, that is

$$|\gamma| = m/s; \quad m = 1, \ldots, s-1$$

then, at each sth point of the set $Q = (j\pi/2)^2$ ($j = s, 2s, 3s, \ldots$), the denominator of the expression being considered (3.12) also vanishes. This also denotes the "collapse" of each sth zone.

The situation becomes clearer when the dependence $p = p(Q; \gamma)$ is considered in three-dimensional space. The corresponding surface really has a cellular "honeycomb" structure and its intersections with the planes $\gamma = 1/5$ and $\gamma = 1/2$ in the positive half-space $p > 0$ are shown in Fig. 2. The geometry of the domain of definition of the expression $p(Q; \gamma)$ provides a representation of the mutual arrangement of the cells in space.

We will initially consider the neighbourhood of one of the points where the zone disappears locally in the $Q\gamma$ plane. We put

$$Q = (s\pi/2)^2(1 + \delta Q), \quad \gamma = m(1 + \delta\gamma)/s \tag{3.14}$$

where $|\delta\gamma| \ll 1$, $|\delta Q| \ll 1$.

On expanding the numerator and denominator of expression (3.12) in powers of δQ, $\delta\gamma$ and retaining only the first non-zero terms, we obtain

$$p = \pm\frac{\pi}{2}\frac{(s^2 - m^2)\delta Q}{[m^2(\delta Q + 2\delta\gamma)^2 - s^2(\delta Q)^2]^{1/2}} \tag{3.15}$$

It is obvious that, on approaching the value $\delta\gamma = 0$ ($\gamma = m/s$), the width of this domain reduces to zero and hence the zone being considered disappears ("collapses"). A fragment of the domain of definition of (3.12) is globally shown in Fig. 4 (the hatched segments). In view of the symmetry, only the upper half-plane for positive values of γ is shown. The sections shown in Fig. 2(a) pass, respectively, along the lines $\gamma = 1/2$ and $\gamma = 1/5$. When the parameter γ is rational, an infinite sequence of points of contraction of the domain of definition appears on the line $\gamma = \text{const}$ in the $Q\gamma$ plane which denotes the collapse of the corresponding sequence of zones.

4. REMARKS ON THE USE OF THE METHOD OF AVERAGING

The transformations described above are suitable for constructing periodic systems, which are only achieved, generally speaking, under certain initial conditions. The solution of the Cauchy problem can

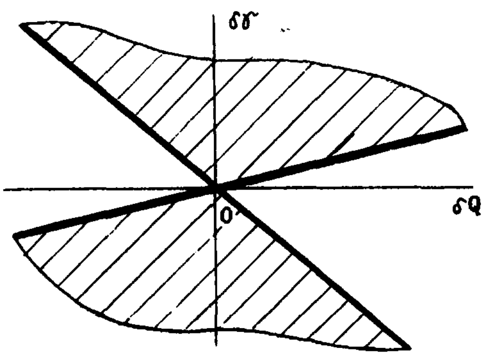

Fig. 3.

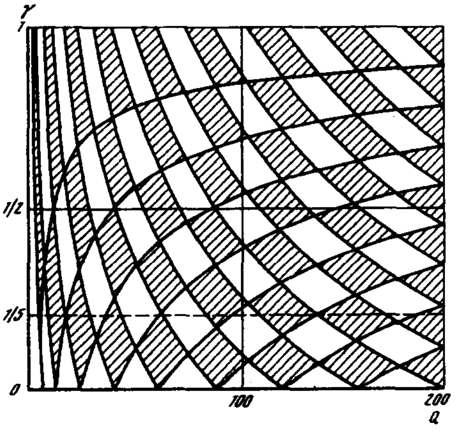

Fig. 4.

be obtained using the idea of averaging. In this case, the formalism of the method of two-scale expansions [8, 13] is convenient. This is attributable to the specific nature of the transformation of (1.2) which is used and associates the new time variable τ with the excitation rate.

We shall assume that the inequality

$$1/\omega \equiv \varepsilon \ll 1 \tag{4.1}$$

holds, that is, that the excitation rate is high in the initial time scale t. We now consider Eq. (2.1), subject to the following initial condition

$$x|_{t=0} = x^\circ \tag{4.2}$$

The solution of problem (2.1), (4.2) is sought in the form

$$x = X(\tau, t^\circ) + Y(\tau, t^\circ)e; \quad \tau = \tau(t/\varepsilon), \quad e = e(t/\varepsilon) \tag{4.3}$$

where the notation $t^\circ \equiv t$ is introduced for convenience. Hence, the required solution is assumed to be dependent on two time variables: a "fast" oscillating time τ and a "slow" time t°.

The subsequent procedure additionally allows the right-hand side of system (2.1) to depend on the variable t°

$$p = p(t^\circ), \quad f = f(x, \varphi, t^\circ)$$

Substituting (4.3) into (2.1), we arrive at a system of partial differential equations

$$\frac{\partial X}{\partial \tau} + \beta\frac{\partial Y}{\partial \tau} = \varepsilon\left(I_f - \frac{\partial Y}{\partial t^\circ}\right), \quad \alpha\frac{\partial Y}{\partial \tau} = \varepsilon\left(R_f - \frac{\partial X}{\partial t^\circ}\right) \tag{4.4}$$

with the conditions

$$Y|_{\tau=\pm 1} = \varepsilon p \tag{4.5}$$

The asymptotic ($\varepsilon \to 0$) solution of problem (4.4), (4.5) can be found using the standard procedure of the method of two-scale expansions [8, 13] in the form of series in powers of the parameter ε.

5. CONCLUSION

A standard pair of non-smooth periodic functions which join small sections of the solution (in the intervals between impulses) into a single analytic expression over the whole time interval plays an important role in the proposed method. In the symmetric case ($\gamma = 0$), the pair of functions which is used $\{\tau, e\}$ is associated in a natural way with the motion of a free point mass between two fixed arresting devices, that is, with one of the two simplest mechanical oscillators. The question involves a vibrational-impact system with a single impact pair and a harmonic oscillator (it is known that the latter generates a pair of trigonometric functions $\{\sin t, \cos t\}$ possessing a number of convenient mathematical properties). Hence, the "joining procedure" which has been proposed also has a certain physical meaning.

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